

Remarks on a solvable cosmological model

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February 7, 2008

Abstract

We present an exact analytical solution of the Einstein equations with cosmological constant in a spatially flat Robertson-Walker metric. This is interpreted as an isotropic Lemaitre-type version of the cosmological Friedmann model. Implications in the recent discovered cosmic acceleration of the universe and in the theory of an inflationary model of the universe are in view. Some properties of this solution are pointed out as a result of numerical investigations of the model.

Pacs 04.62.+v

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1 Introduction

Recent astrophysical investigations [2, 3] demonstrate that the expansion of the universe is accelerating rather than slowing down. This may change our picture of the universe suggesting that we live in an accelerating and flat universe. However, from the theoretical point of view intensive efforts are done [4] in order to accommodate the relativistic cosmology with an accelerating universe. One of the major results of this situation is the fact that the most popular solution (inspired by what is happening in the theory of inflationary cosmology) is to consider models which satisfy Einstein equations with cosmological constant in order to induce cosmic acceleration. Actually, several models until now not being in the main stream of the modern cosmology, are again in view of cosmologists. We believe that it is necessary to more carefully investigate new possible versions of the the Friedman-Robertson-Walker (FRW) model [5, 6] in order to see how may we implement here cosmic acceleration without loosing the well-known results of the model in describing early stages of the universe.

Another important problem is to try to find simple analytically solvable models which may be easily interpreted from the physical point of view. Moreover, it is clear that many further developments of the more complicated models have to be done using algebraic or numerical methods on computers. Then it is important to have at least one simple analytically solvable model for testing the computational methods and verify their degree of confidence.

This article is dedicated to the revealing of an analytical solution of the Einstein equations with cosmological constant in a version of the FRW model with flat space metric [5]. We present some properties of this solutions and the possible implications to the more accurate description of the inflationary and accelerated universe.

We must specify that the model with cosmological constant and $k = 1$ is known in literature as Lemaitre-FRW model but, unfortunately, this does not have analytical solution. In other respects, we believe that the solution for $k = 0$ we present here is ignored by the investigators of this cosmology since they are forced to find solutions of the Einstein equations in terms of the measurable quantities (as the Hubble constant/parameter, the redshift or the deceleration parameter) and searching for an approximate solution for the scale factor of the universe as a Taylor expansion in terms of these quantities. Thus the existence of an analytical solution is hidden by the large

number of solutions available only for restricted periods from the history of universe.

2 The model and its solution

Let us consider the Lemaitre-type version of the FRW model with flat space (i.e., $k = 0$) [5], positively defined cosmological constant, Λ , and linear dependence between the density of energy, ϵ , and the pressure, p . In general, in an arbitrary local chart (or natural frame) of coordinates x^μ ($\mu, \nu, \dots = 0, 1, 2, 3$), the Einstein equations of this model (written in natural units with $c = 1$),

$$R_\mu^\nu - \frac{1}{2}R\delta_\mu^\nu - \Lambda\delta_\mu^\nu = 8\pi GT_\mu^\nu, \quad (1)$$

involve the usual classical stress-energy tensor

$$T_\mu^\nu = (\epsilon + p)u^\nu u_\mu - p\delta_\mu^\nu, \quad (2)$$

which depends on the covariant four-velocity $u^\mu = dx^\mu/ds$.

In the preferred frame with Cartesian coordinates $x^0 = t$, \vec{x} which has the usual FRW line element with $k = 0$,

$$ds^2 = dt^2 - a(t)^2 d\vec{x}^2, \quad (3)$$

we have $u^0 = u_0 = 1$ and $u^i = 0$ ($i = 1, 2, 3$) such that the Einstein equations reduce to the simple system

$$3\left(\frac{\dot{a}}{a}\right)^2 = \Lambda + 8\pi G \epsilon \quad (4)$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = \Lambda - 8\pi G p \quad (5)$$

where we denoted $\dot{} = d/dt$. Assuming that ϵ and p are functions only on t and satisfy a linear equation of state, namely

$$p = \kappa\epsilon, \quad (6)$$

one can integrate this system with the conditions

$$\epsilon(0) = \epsilon_0, \quad a(0) = 1, \quad (7)$$

at the present time $t = 0$. Indeed, (4), (5) and (6) are equivalent with the equation

$$\frac{d}{dt} \left(\frac{\dot{a}}{a} \right) = \frac{\kappa + 1}{2} \left[\Lambda - 3 \left(\frac{\dot{a}}{a} \right)^2 \right] \quad (8)$$

which, after a little calculation and exploiting (7), leads to the final solution

$$a(t) = \left[\cosh \frac{\sqrt{3\Lambda}}{2} (\kappa + 1) t + \sqrt{1 + \frac{8\pi G \epsilon_0}{\Lambda}} \sinh \frac{\sqrt{3\Lambda}}{2} (\kappa + 1) t \right]^{2/3(\kappa+1)} \quad (9)$$

while the density of energy reads

$$\epsilon(t) = \epsilon_0 a(t)^{-3(\kappa+1)}. \quad (10)$$

Finally, we calculate the Bang time, $-t_0$, from the initial condition $a(-t_0) = 0$ as

$$t_0 = \frac{2}{\sqrt{3\Lambda} (\kappa + 1)} \operatorname{argcoth} \sqrt{1 + \frac{8\pi G \epsilon_0}{\Lambda}}, \quad (11)$$

completing thus the solution of this simple model.

Let us observe that the conservation law $T^\mu_{\nu;\mu} = 0$ is implicitly fulfilled since (10) is simultaneously the solution of this conservation equation which takes the form

$$\dot{\epsilon} + 3\epsilon(\kappa + 1) \frac{\dot{a}}{a} = 0 \quad (12)$$

when the equation of state (6) is accomplished. Moreover, it is important that the equation (10) is independent on Λ since then it holds even in the particular case of $\Lambda \rightarrow 0$ when we recover the usual solution of the FRW model with $k = 0$ [5, 6],

$$a(t) = \left[1 + \sqrt{6\pi G \epsilon_0} (\kappa + 1) t \right]^{2/3(\kappa+1)}. \quad (13)$$

The interesting new features of our model are due to the cosmological constant which gives an hyperbolic character to the general model or to its asymptotic behavior. It is clear that the universe devoid of matter becomes

a de Sitter one but it is remarkable that in the far future the time behavior of $a(t)$ is also of the de Sitter type since for very large t we have

$$a(t) \sim \alpha e^{\sqrt{\frac{\Lambda}{3}}t}, \quad \alpha = \left(\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{8\pi G\epsilon_0}{\Lambda}} \right)^{2/3(\kappa+1)}. \quad (14)$$

In other words if we rescale the Cartesian space coordinates with the factor α then the line element (3) becomes just the de Sitter one in the limit of $t \rightarrow \infty$. Thus we can conclude that our solution is compatible with the well-known models describing the Big-Bang scenario and for late behavior it shows off characteristics of a de Sitter spacetime.

Many physical effects depend on the form of the Hubble function that in our model reads

$$H(t) = \frac{\dot{a}}{a} = \frac{1}{\sqrt{3(\Lambda + 8\pi G\epsilon_0)}} \left[\Lambda + \frac{8\pi G\epsilon_0}{1 + \sqrt{1 + \frac{8\pi G\epsilon_0}{\Lambda}} \tanh \frac{\sqrt{3\Lambda}}{2}(\kappa + 1)t} \right]. \quad (15)$$

Obviously, in the late time limit we have

$$\lim_{t \rightarrow \infty} H(t) = \sqrt{\frac{\Lambda}{3}}$$

while the actual value of the Hubble function (i.e., the Hubble constant),

$$H_0 = H(t) |_{t=0} = \sqrt{\frac{\Lambda + 8\pi G\epsilon_0}{3}}, \quad (16)$$

depends on the actual value of the density of energy, ϵ_0 , and Λ .

3 Numerical investigations

This section is dedicated to several numerical results we obtained using our analytical solution which reveals some interesting features of the model. First we have plotted the behavior of the scale factor in time for a constant value of the cosmological constant Λ . In this purpose we considered an accepted value of the actual mass-density of the universe as $\rho_0 = 3 \cdot 10^{-28} \text{Kg}/\text{m}^3$ (in IS units where $\epsilon_0 = \rho_0 c^2$) and of the actual Hubble constant as $H_0 =$

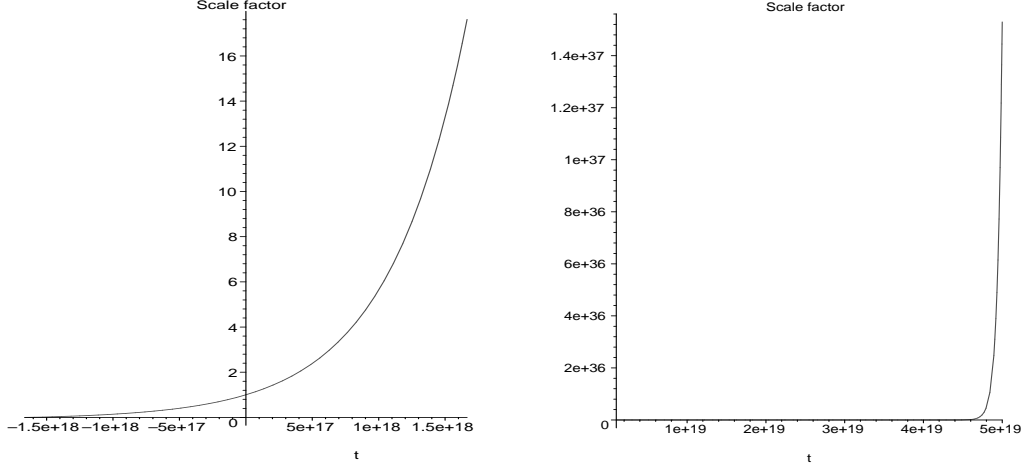


Figure 1: Evolution of the scale factor $a(t)$ from the Big-Bang (left panel) and the fast growing of the scale factor at late time $t = 10^{18} \dots 5 \cdot 10^{19} \text{ s}$ (right panel)

$(1.8 \cdot 10^{10} \text{ years})^{-1}$ [8]. Then, using (16) we can estimate the cosmological constant as being $\Lambda \approx 0.977 \cdot 10^{-52} \text{ m}^{-2}$ which will be the value used in the next numerical investigations. We also choose $\kappa = -1/2$. Two first graphs are showed in Figure 1 for different intervals of time, one before the actual time $t = 0$ and the other one for late behavior of the universe. It is obvious, especially from the second picture (right panel) the fast growing of the scale factor for late time, showing the accelerated stage of the universe described by this model. We have to mention that we estimated the Big-Bang time t_0 using the above numerical values and formula (11). Thus we have used in our figures the resulting value $t_0 \approx 0.1665 \cdot 10^{19} \text{ s}$ (≈ 52.7 billion years). This value is larger than the recognized one and appears here due to the small value of the mass-density we used. A greater value for ρ , including dark matter and other unknown components, (for example one ten times larger) gives a Big-Bang time $t_0 \approx 29$ billion years. .

Figure 2 presents the time behavior of the Hubble constant (the left panel) as defined in (15) for the same parameters as for the above graphs. As we pointed out in the comments ending the last section, the Hubble function is evolving to a constant value $\sqrt{\Lambda/3}$ specific for the de Sitter model. This conclusion is coupled to the late time behavior of the scale factor (see above)

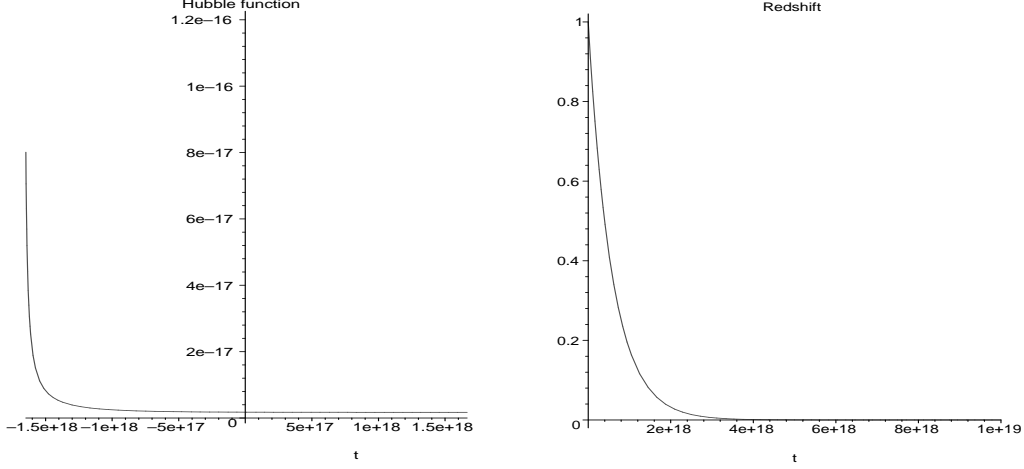


Figure 2: Evolution of the Hubble function (left panel) and of the redshift defined as $1/a(t)$ (right panel)

which is also specific to a de Sitter metric. In the right panel of the same Figure 2 we plotted the redshift defined as $1/a(t)$.

Next plots, Figure 3, are dedicated to the time behavior of the acceleration defined here as the second time derivative of the scale factor, $\ddot{a}(t)$. Before the actual time $t = 0$ (presented in the left panel) shows the decreasing of the acceleration after the Big-Bang at $t \geq -t_0$. In the right panel we observe the increasing of the acceleration at later time in the universe evolution. Thus we can conclude that it will be possible to use our solution for modeling the so-called “cosmic-acceleration” in discussion in the modern astrophysics.

Last of this type of plots, Figure 4, represent the time behavior of the density function $\epsilon(t)$ given by equation (10) and of the deceleration defined as $\ddot{a}(t)/[a(t)H(t)^2]$ [5, 8].

Acknowledgments

One of the authors (D. N. V.) is deeply indebted to Prof. B. F. Schutz and Prof. E. Seidel for several invitations at the Albert Einstein Institute, Golm, Germany where partially this research was done. Many thanks also for fruitful discussions on the subject with M. Alcubierre and F. Guzman.

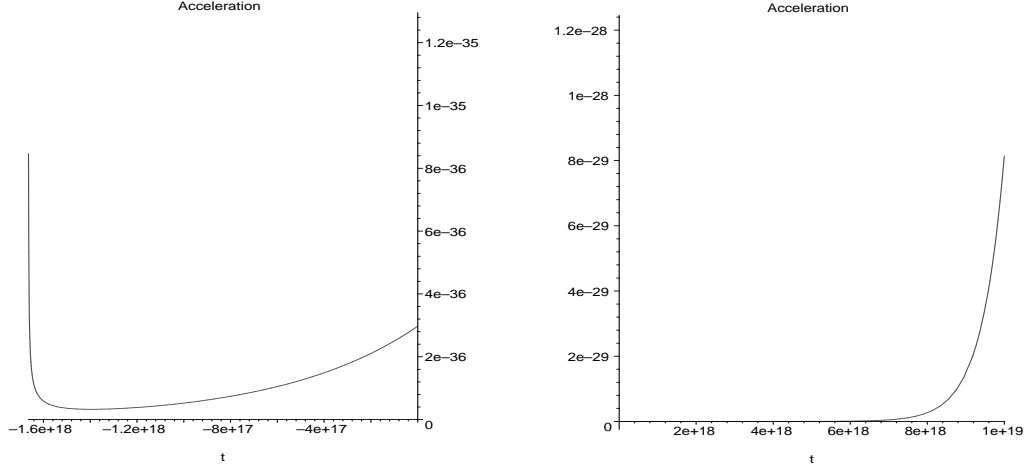


Figure 3: Evolution of the acceleration, $\ddot{a}(t)$, before the present time (left panel) and after $t = 0$ (right panel)

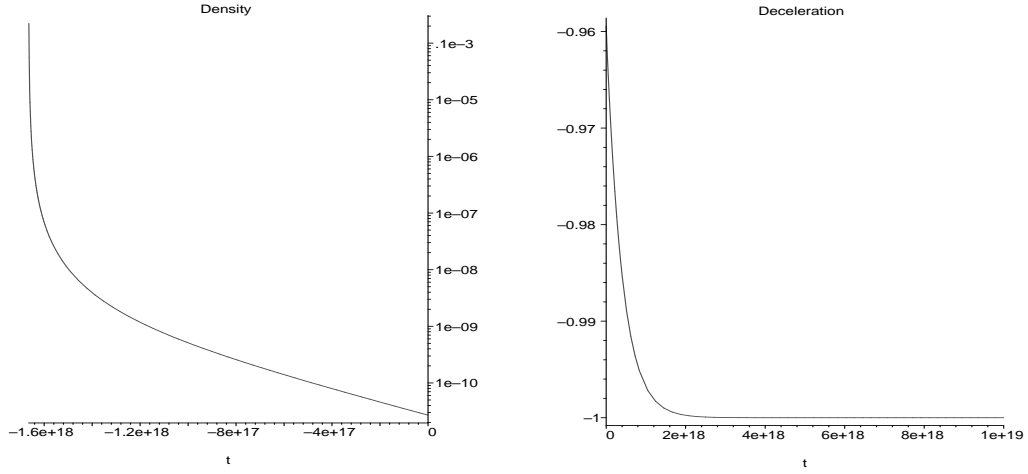


Figure 4: Evolution of the density before the actual time $t = 0$ (left panel) and of the deceleration, $\ddot{a}(t)/[a(t)H(t)^2]$, after the actual time (right panel)

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